Recall:

定理(Frobenius 可积性定理)　　(L 称 l 维分布)

在 p 邻域 U 内, $L = \mathrm{Span}\{X_1, \cdots, X_l\}$, X_i 线性无关, 则

存在 p 处局部坐标 (W, w^i), s.t. $X_i = \frac{\partial}{\partial w^i}$, $i = 1, \cdots, l$

$\Leftrightarrow \forall i, j, [X_i, X_j] = 0$

定义: 分布的积分子流形

L 是 N 上 l 维分布, 称 L 完全可积, 若存在 N 的浸入子流形 M,

s.t. $T_p M = L_p$, $\forall p \in M$

注: 1 维分布对应的积分子流形即为积分曲线.

定理: 完全可积 $\Leftrightarrow$ 对合

例. $\mathbb{R}^{n+1}$, $X_i = \frac{\partial}{\partial x^i} + a^i(x^1, \cdots, x^n) \frac{\partial}{\partial x^{n+1}}$, $i = 1, 2, \cdots, n$

求 $\mathrm{Span}\{X_i\}$ 完全可积的条件

解: $[X_i, X_j] f = \left(\frac{\partial}{\partial x^i} + a^i \frac{\partial}{\partial x^{n+1}}\right)\left(\frac{\partial f}{\partial x^j} + a^j \frac{\partial f}{\partial x^{n+1}}\right)$

$- \left(\frac{\partial}{\partial x^j} + a^j \frac{\partial}{\partial x^{n+1}}\right)\left(\frac{\partial f}{\partial x^i} + a^i \frac{\partial f}{\partial x^{n+1}}\right)$

$= \frac{\partial^2 f}{\partial x^i \partial x^j} + a^i \frac{\partial^2 f}{\partial x^{n+1} \partial x^j} + \frac{\partial a^j}{\partial x^i} \frac{\partial f}{\partial x^{n+1}} + a^j \frac{\partial^2 f}{\partial x^i \partial x^{n+1}} + a^i a^j \frac{\partial^2 f}{(\partial x^{n+1})^2}$

$- \sim$

$= \left(\frac{\partial a^j}{\partial x^i} - \frac{\partial a^i}{\partial x^j}\right) \frac{\partial}{\partial x^{n+1}} f$

故 $[X_i, X_j] = \left(\frac{\partial a^j}{\partial x^i} - \frac{\partial a^i}{\partial x^j}\right) \frac{\partial}{\partial x^{n+1}} \in \mathrm{Span}\{X_1 \cdots X_n\}$

$\Rightarrow \frac{\partial a^j}{\partial x^i} = \frac{\partial a^i}{\partial x^j}$, $\forall i, j$

思考：如何求积分子流形？

因 $[X_i, X_j]=0$, $\forall i,j$，故存在局部坐标 $t^1, \cdots t^{n+1}$, s.t.

$X_i = \frac{\partial}{\partial t^i}$，$i=1,2,\cdots,n$. 从而积分子流形即为 $\{(t^1,\cdots,t^{n+1}) \mid t^{n+1}=c\}$

于是问题转化为，求这样的 $t^1, \cdots t^{n+1}$，

$$X_i = \frac{\partial}{\partial t^i} = \frac{\partial x^j}{\partial t^i}\frac{\partial}{\partial x^j} \Rightarrow \frac{\partial x^i}{\partial t^i}=1, \quad \frac{\partial x^j}{\partial t^i}=0, \quad \frac{\partial x^{n+1}}{\partial t^i}=a^i \quad (j\neq n+1,\ i\leq n)$$

直接取 $x^i=t^i$，$x^{n+1}=t^{n+1}+f(t^1,\cdots,t^n)$

其中，f 使得 $\frac{\partial f}{\partial t^i}=a^i$，$i=1,\cdots,n$

命题：$f=\sum_{i=1}^{n}\int_0^1 a^i(t\cdot t^1, t\cdot t^2,\cdots,t\cdot t^n)\,dt\cdot t^i + c$

于是我们求得，积分子流形

$\{t^{n+1}=c'\} = \{(x^1,\cdots,x^n, c+\sum_{i=1}^{n}\int_0^1 a^i(t\cdot t^1, t\cdot t^2,\cdots,t\cdot t^n)\,dt\cdot t^i\}$

命题的验证：

$$\begin{aligned}\frac{\partial f}{\partial t^j} &= \sum_{i=1}^{n}\int_0^1 \partial_j a^i(t\cdot t^1,\cdots t\cdot t^n)\cdot t\,dt\cdot t^i + \int_0^1 a^j(t\cdot t^1,\cdots t\cdot t^n)\,dt \\ &= \int_0^1 \sum_{i=1}^{n}\partial_i a^j(t\cdot t^1,\cdots t\cdot t^n)\cdot t\,dt\cdot t^i + \int_0^1 a^j(t\cdot t^1,\cdots t\cdot t^n)\,dt \\ &= \int_0^1 \frac{d}{dt}a^j(t\cdot t^1,\cdots t\cdot t^n)\cdot t\,dt + \int_0^1 a^j(t\cdot t^1,\cdots t\cdot t^n)\,dt \\ &= a^j(t^1,\cdots,t^n)\end{aligned}$$

注：求解积分子流形的过程实际上是一个解方程的过程，我们实际上得到了一个微分方程的定理

定理：$\frac{\partial f}{\partial x^i}=a^i$，$i=1,\cdots,n$ 有解 $\Longleftrightarrow$ $\frac{\partial a^i}{\partial x^j}=\frac{\partial a^j}{\partial x^i}$

我们再证明一个更一般性的命题

定理：$U\times V \underset{open}{\subset} \mathbb{R}^m\times\mathbb{R}^n$， $f_i^\alpha : U\times V \to \mathbb{R}$ 光滑，$1\le i\le m$，$1\le\alpha\le n$

则对于 $\forall\,(x_0,y_0)\in U\times V$，方程组

$$(*)\quad \begin{cases} \dfrac{\partial y^\alpha}{\partial x^i} = f_i^\alpha(x,y) \\ y^\alpha(x_0) = y_0^\alpha \end{cases}$$

在 x_0 某邻域 $W\subset U$ 内有唯一解 $y^\alpha = y^\alpha(x)$，$x\in W$ $\Leftrightarrow$

$$\frac{\partial f_i^\alpha}{\partial x^j} + \sum_{\beta=1}^n \frac{\partial f_i^\alpha}{\partial y^\beta} f_j^\beta = \frac{\partial f_j^\alpha}{\partial x^i} + \sum_{\beta=1}^n \frac{\partial f_j^\alpha}{\partial y^\beta} f_i^\beta$$

Rmk：这个条件的得来十分自然，在证明必要性时可看出

证明："$\Rightarrow$" $\dfrac{\partial^2 y^\alpha}{\partial x^i\partial x^j} = \dfrac{\partial}{\partial x^i} f_j^\alpha(x,y) = \dfrac{\partial}{\partial x^i} f_j^\alpha + \displaystyle\sum_{\beta=1}^n \frac{\partial f_j^\alpha}{\partial y^\beta}\frac{\partial y^\beta}{\partial x^i}$

$$= \frac{\partial f_j^\alpha}{\partial x^i} + \sum_{\beta=1}^n \frac{\partial f_j^\alpha}{\partial y^\beta} f_i^\beta$$

$\dfrac{\partial^2 y^\alpha}{\partial x^i\partial x^j} = \dfrac{\partial^2 y^\alpha}{\partial x^j\partial x^i}$ 即可得必要性

"$\Leftarrow$" 不失一般性，$x_0 = 0\in U$，

Step 1：确定函数 $y_1^\alpha(x^1,0,\cdots,0)$，s.t

$$\begin{cases} \dfrac{\partial y_1^\alpha(x^1,0,\cdots,0)}{\partial x^1} = f_1^\alpha(x^1,0,\cdots,0,\,y_1(x^1,0,\cdots,0)) \\ y_1^\alpha(0,\cdots,0) = y_0^\alpha \end{cases}$$

此时，$y_1 = (y_1^1,\cdots y_1^n)$ 是一个只与 x^1 相关的函数，问题转化为

$\xi(t) = (\xi^1(t),\cdots \xi^n(t))$，s.t

$$(\Delta)\quad \begin{cases} \dfrac{d\xi^\alpha}{dt} = g^\alpha(t,\xi) \\ \xi(0) = \xi_0 \end{cases}$$

$\Rightarrow$ 在某个小区间 $|t|<\varepsilon_1$ 内有解 $\Rightarrow$ $y_1(x^1,0,\cdots,0)$ 存在

Step 2：固定 x^1，找到函数 $y_2^\alpha(x^1,x^2,0,\cdots 0)$，s.t.

$$\begin{cases} \dfrac{\partial y_2^\alpha(x^1,x^2,0,\cdots,0)}{\partial x^2} = f_2^\alpha(x^1,x^2,0,\cdots,0,y_2^\alpha(x^1,x^2,0\cdots 0)) \\ y_2^\alpha(x^1,0,\cdots,0) = y_1^\alpha(x^1,0,0,\cdots) \end{cases}$$

注意，这里 x^1 已固定，方程变量只有 x^2，转化为 (△)

于是，我们得到 $y_2(x^1,x^2,0,\cdots,0)$

显然地，y_2 关于 x^2 的偏导满足

$$\frac{\partial y_2^\alpha}{\partial x^2} = f_2^\alpha(x^1,x^2,0,\cdots,0,y_2)$$

那么关于 x^1 的偏导呢？这里就会用到定理条件了。

令 $g^\alpha(x^1,x^2) = \dfrac{\partial y_2^\alpha(x^1,x^2,0,\cdots,0)}{\partial x^1} - f_1^\alpha(x^1,x^2,0,\cdots,0,y_2(x^1,x^2,0,\cdots,0))$

由条件 $y_2^\alpha(x^1,0,\cdots,0) = y_1^\alpha(x^1,0,\cdots,0)$，可知

$$g^\alpha(x^1,0) = 0 \quad ,\forall x^1$$

$$\begin{aligned}
\frac{\partial}{\partial x^2} g^\alpha(x^1,x^2) &= \frac{\partial}{\partial x^1}\left(\frac{\partial}{\partial x^2} y_2^\alpha(x^1,x^2,0,\cdots,0)\right) - \frac{\partial}{\partial x^2} f_1^\alpha(x^1,x^2,0\cdots 0,y_2(x^1,x^2,0\cdots 0)) \\
&= \frac{\partial f_2^\alpha}{\partial x^1} + \frac{\partial f_2^\alpha}{\partial y^\beta}\underbrace{\frac{\partial y_2^\beta(x^1,x^2,0,\cdots,0)}{\partial x^1}}_{g^\beta+f_1^\beta} - \frac{\partial f_1^\alpha}{\partial x^2} - \frac{\partial f_1^\alpha}{\partial y^\beta}\underbrace{\frac{\partial y_2^\beta(x^1,x^2,0,\cdots,0)}{\partial x^2}}_{=f_2^\beta} \\
&= \frac{\partial f_2^\alpha}{\partial y^\beta} g^\beta + \frac{\partial f_2^\alpha}{\partial x^1} + \frac{\partial f_2^\alpha}{\partial y^\beta} f_1^\beta - \frac{\partial f_1^\alpha}{\partial x^2} - \frac{\partial f_1^\alpha}{\partial y^\beta} f_2^\beta \\
&= \frac{\partial f_2^\alpha}{\partial y^\beta} g^\beta
\end{aligned}$$

因此，固定 x^1，我们得到 $\begin{cases} \dfrac{\partial g}{\partial x^2} = \sum\limits_\beta \dfrac{\partial f_2^\alpha}{\partial y^\beta} g^\beta(x^1,x^2) \\ g\big|_{x^2=0} = 0 \end{cases}$

为一阶线性方程，由解的唯一性 $\Rightarrow g(x^1,x^2) \equiv 0$

这个定理十分常用，在微分相关的课程都会出现，如微分几何、微分流形，但一般会不加证明地使用

从微分形式看积分子流形

对于局部坐标 $(x^1,\cdots,x^n)$，$L = \mathrm{Span}\{\frac{\partial}{\partial x^1},\cdots \frac{\partial}{\partial x^k}\}$，显然 L 对应积分子流形 $\mathbb{R}^k$，$\frac{\partial}{\partial x^1},\cdots \frac{\partial}{\partial x^n}$ 作为 TM 的基有对偶基 $\omega^1,\cdots,\omega^n$，于是

$L = \{X\in TM \mid \omega^\alpha X = 0,\ \alpha = k+1,\cdots,n\}$

L 完全可积 $\Leftrightarrow$ $\forall X,Y\in L$, $[X,Y]\in L$ $\Leftrightarrow$ $\omega^\alpha([X,Y])=0$

$$d\omega^\alpha(X,Y) = X\omega^\alpha(Y) - Y\omega^\alpha(X) - \omega^\alpha([X,Y]) = 0$$

作为2-form，$d\omega^\alpha = \sum_{i<j} C^\alpha_{ij}\,\omega^i\wedge\omega^j$，若存在 $j\le k$，$C^\alpha_{ij}\neq 0$

则 $d\omega^\alpha(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}) = C^\alpha_{ij}\neq 0$，矛盾。故 $d\omega^\alpha = \sum_{j=k+1}^{n}(\sum_{i<j} C^\alpha_{ij}\,\omega^i)\wedge\omega^j$

当然，这只是一种特殊情况，但我们可将其推广：

定理：设 $\omega^1\dots\omega^k$ 为 k 个 1-form，$L = \{X \mid \omega^i(X)=0,\ i=1,\cdots,k\}$

则 L 完全可积 $\Leftrightarrow$ $\exists$ k^2 个 1-form θ^i_j，s.t. $d\omega_j = \sum_i \theta^i_j\wedge\omega_i$

四、设$u(x)\in C^\infty(\mathbb{R}^n)$, (a)求Pfaff方程组$\sum_{k=1}^n \frac{\partial u}{\partial x^k}dx^k - dx^{n+1} = 0$在$\mathbb{R}^{n+1}$中所确定的分布$L$, (b)判断其可积性, (c)若$L$可积求出其积分子流形.

解. (a) 记$\omega = \sum_{k=1}^n \frac{\partial u}{\partial x^k}dx^k - dx^{n+1}$. 由定义, $L_p = \{X\in T_p\mathbb{R}^{n+1}|\omega(X)=0\}$为$T_p\mathbb{R}^{n+1}$的$n$维子空间. 直接计算得, $\{\frac{\partial}{\partial x^k} + \frac{\partial u}{\partial x^k}\frac{\partial}{\partial x^{n+1}}\}_{k=1}^n$都是$L_p$中的向量. 由

$$\mathrm{rank}\begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ \frac{\partial u}{\partial x^1} & \cdots & \frac{\partial u}{\partial x^n} \end{bmatrix} = n$$

知$\{\frac{\partial}{\partial x^k} + \frac{\partial u}{\partial x^k}\frac{\partial}{\partial x^{n+1}}\}_{k=1}^n$是线性无关向量组. 于是$L_p = \mathrm{span}\{\frac{\partial}{\partial x^k} + \frac{\partial u}{\partial x^k}\frac{\partial}{\partial x^{n+1}}\}_{k=1}^n$.

(b) 由$d\omega = \sum_{k,l}\frac{\partial u}{\partial x^k x^l}dx^k\wedge dx^l = 0$知此Pfaff方程组确定的分布可积.

(c) 对任意$c\in\mathbb{R}$, $S_c := \{(x^1,\dots,x^n,u(x)+c)|(x^1,\dots,x^n)\in\mathbb{R}^n\}$是正则光滑映射$f:\mathbb{R}^{n+1}\to\mathbb{R}$, $(x^1,\dots,x^n,x^{n+1})\mapsto u(x^1,\dots,x^n) = x^{n+1}$的水平集, 进而是$\mathbb{R}^{n+1}$的嵌入子流形. 注意到$df = \omega$, 对任意$p\in S_c$, 我们有$T_pS_c = \{X\in T_p\mathbb{R}^{n+1}|0 = df(X) = \omega(X)\} = L_p$, 故$S_c$是$L$的积分子流形. 因为$\bigcup_{c\in\mathbb{R}} S_c = \mathbb{R}^{n+1}$, 所以$\{S_c\}_{c\in\mathbb{R}}$是$L$的所有(极大)积分子流形. □

3. Frobenius定理的对偶形式

设M为n维流形，$U\overset{开}{\subset}M$，$X_1,\cdots,X_l$为U上处处线性无关的向量场。$L=\mathrm{Span}\{X_1,\cdots,X_l\}$是$U$上$l$维的分布。补上$X_{l+1},\cdots,X_n$使得$X_1,\cdots,X_l,X_{l+1},\cdots,X_n$构成$U$上的一组基向量场，其对偶记为$\omega^1,\cdots,\omega^n$。由对偶性知，分布$L$等价于Pfaff方程组$\{\omega^\alpha=0 \mid \alpha=l+1,\cdots,n\}$，即$L=\{X\in T_pM \mid p\in U,\ \omega^\alpha(X)=0,\ \alpha=l+1,\cdots,n\}$.

定理. 光滑分布L对合 $\Longleftrightarrow$ 存在$(n-l)^2$个1次微分式θ^α_β，使得

$$d\omega^\alpha=\sum_{\beta=l+1}^{n}\theta^\alpha_\beta\wedge\omega^\beta,\ 或\ d\omega^\alpha\equiv 0,\ \mathrm{mod}\ \omega^{l+1},\cdots,\omega^n.$$

证明. 约定：$1\le A,B,C,\cdots\le n$, $1\le i,j,k,\cdots\le l$, $l+1\le\alpha,\beta,\gamma,\cdots\le n$.

由于$\omega^1,\cdots,\omega^n$是一组基，于是可设

$$d\omega^\alpha=\sum_{i,j}C^\alpha_{ij}\,\omega^i\wedge\omega^j+2\sum_{i,\beta}C^\alpha_{i\beta}\,\omega^i\wedge\omega^\beta+\sum_{\beta,\gamma}C^\alpha_{\beta\gamma}\,\omega^\beta\wedge\omega^\gamma,$$

其中$C^\alpha_{AB}=-C^\alpha_{BA}$. 由上式知：

$$d\omega^\alpha(X_i,X_j)=\Big(\sum_{i,j}C^\alpha_{ij}\,\omega^i\wedge\omega^j\Big)(X_i,X_j)=2C^\alpha_{ij}. \quad \text{—— (1)}$$

又，
$$d\omega^\alpha(X_i,X_j)=X_i(\omega^\alpha(X_j))-X_j(\omega^\alpha(X_i))-\omega^\alpha([X_i,X_j])$$
$$=-\omega^\alpha([X_i,X_j]) \quad \text{—— (2)}$$

$(\Longrightarrow)$ 由L对合，知$[X_i,X_j]=\sum_{k=1}^{l}C^k_{ij}X_k$. 由(1),(2)知$C^\alpha_{ij}=0$.

于是，$d\omega^\alpha=2\sum_{i,\beta}C^\alpha_{i\beta}\,\omega^i\wedge\omega^\beta+\sum_{\beta,\gamma}C^\alpha_{\gamma\beta}\,\omega^\gamma\wedge\omega^\beta=\sum_{\beta=l+1}^{n}\theta^\alpha_\beta\wedge\omega^\beta$, $\theta^\alpha_\beta=2C^\alpha_{i\beta}\omega^i+C^\alpha_{\gamma\beta}\omega^\gamma$.

$(\Longleftarrow)$ 设$[X_i,X_j]=\sum_{k=1}^{l}a^k_{ij}X_k+\sum_{\alpha=l+1}^{n}a^\alpha_{ij}X_\alpha$. 由$d\omega^\alpha=\sum_{\beta=l+1}^{n}\theta^\alpha_\beta\wedge\omega^\beta$知

$$0=d\omega^\alpha(X_i,X_j)=-\omega^\alpha([X_i,X_j])=-a^\alpha_{ij}.$$ 因此，$[X_i,X_j]=\sum_{k=1}^{l}a^k_{ij}X_k$. #